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On Systems of Linear Differential Equations of the First Order.*

BY MAXIME BÔCHER.

It is my object in this paper to consider the system of n differential equations:

$$\frac{dy_i}{dx} = \sum_{j=1}^{j=n} \alpha_{ij} y_j + \beta_i, \quad (i = 1, 2, \dots n), \quad (1)$$

where the coefficients α and β are functions, not necessarily analytic, of the real variable x which, throughout the interval

$$(I) \quad a \leq x \leq b,$$

satisfy the following condition:

A. *The functions α_{ij} and β_i shall have at most a finite number of discontinuities in (I) , and the integrals*

$$\int |\alpha_{ij}| dx, \quad \int \beta_i dx$$

shall converge, when extended over any portion of (I) .

We add the definition:

B. *A set of n functions of x ($y_1, y_2, \dots y_n$) are said to form a solution of (1) when, and only when, they are all of them continuous at every point of (I) , and, except at most at a finite number of points of this interval, have derivatives which satisfy (1).*

The first theorem we wish to prove is the following:

I. *Any set of n constants ($c_1, c_2, \dots c_n$), and any point c of (I) being given, there exists one, and only one, solution ($y_1, y_2, \dots y_n$) of (1) which satisfies the initial conditions:*

$$y_i(c) = c_i, \quad (i = 1, 2, \dots n). \quad (2)$$

* This paper was read before the American Mathematical Society, February 22, 1902.

To prove that at least one such solution exists, we use the method of successive approximations.* For this purpose consider the series

$$y_i^{(0)} + y_i^{(1)} + y_i^{(2)} + \dots \quad (i = 1, 2, \dots, n), \quad (3)$$

where

$$y_i^{(0)} = c_i + \int_c^x \beta_i dx, \quad (i = 1, 2, \dots, n), \quad (4)$$

and, when $k > 0$,

$$y_i^{(k)} = \int_c^x \left(\sum_{j=1}^n \alpha_{ij} y_j^{(k-1)} \right) dx.$$

It is clear that the integrals here written all converge, and represent functions which are continuous throughout (I) .

In order to prove the convergence of the series (3), let M be a positive constant such that

$$|y_i^{(0)}| \leq M, \quad (i = 1, 2, \dots, n), \quad (5)$$

and let $\phi(x)$ be a function which has at most a finite number of discontinuities in (I) , for which the integral

$$\int \phi(x) dx$$

converges when extended over any portion of (I) , and such that

$$|\alpha_{ij}| \leq \phi(x), \dagger \quad (i, j = 1, 2, \dots, n). \quad (6)$$

We can now establish the fundamental inequalities:

$$|y_i^{(k)}| \leq M \frac{n^k \left| \int_c^x \phi(x) dx \right|^k}{k!}, \quad \begin{matrix} (i = 1, 2, \dots, n) \\ (k = 0, 1, \dots) \end{matrix}. \quad (7)$$

These inequalities reduce to (5) when $k = 0$. We shall therefore have established their truth by the method of mathematical induction if, assuming (7) to

* This method was used for this purpose for the first time by Peano; cf. *Mathematische Annalen*, vol. 32 (1888), p. 450. Peano, however, restricts himself not merely to homogeneous equations (i. e. assumes $\beta_1 \equiv \beta_2 \equiv \dots \equiv \beta_n \equiv 0$), but also restricts the coefficients α_{ij} to be continuous throughout (I) .

† That such a function exists will be seen from the example:

$$\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} |\alpha_{ij}|.$$

hold when $k \leq k_1$, we can prove that it holds when $k = k_1 + 1$. Now we have :

$$|y_i^{(k_1+1)}| \leq \int_c^x \left(\sum_{j=1}^{j=n} |\alpha_{ij}| \cdot |y_j^{(k_1)}| \right) dx \leq M \frac{n^{k_1+1}}{k_1!} \left| \int_c^x \phi(x) \left(\int_c^x \phi(x) dx \right)^{k_1} dx \right|. \quad (8)$$

By integrating by parts we find

$$\int_c^x \phi(x) \left(\int_c^x \phi(x) dx \right)^{k_1} dx = \frac{\left(\int_c^x \phi(x) dx \right)^{k_1+1}}{k_1+1}. \quad (9)$$

The substitution of (9) in (8) gives us (7) for the case $k = k_1 + 1$.

If, now, we write

$$N = \int_a^b \phi(x) dx, \quad (10)$$

we deduce at once from (7) the further inequality :

$$|y_i^{(k)}| \leq M \frac{(nN)^k}{k!}, \quad \left(\begin{matrix} i = 1, 2, \dots, n \\ k = 0, 1, \dots \end{matrix} \right). \quad (11)$$

If, then, we compare (3) with the convergent series of positive constant terms,

$$\sum_{k=0}^{k=\infty} M \frac{(nN)^k}{k!}, \quad (12)$$

we see that the series (3) are absolutely and uniformly convergent throughout (I). Writing

$$y_i = y_i^{(0)} + y_i^{(1)} + y_i^{(2)} + \dots, \quad (i = 1, 2, \dots, n) \quad (13)$$

we see that the functions (y_1, y_2, \dots, y_n) thus defined are continuous throughout (I). Moreover, since, when $x = c$, all the terms except the first of these series vanish, while $y_i^{(0)}(c) = c_i$, it follows that $y_i(c) = c_i$.

In order to prove that (y_1, \dots, y_n) form a solution of (1) according to the definition (B), it will then be sufficient if we can prove that, x_0 being any point where the coefficients α_{ij} and β_i are continuous, (y_1, \dots, y_n) have derivatives at x_0 which satisfy (1).

For this purpose let us surround x_0 by a short interval (J) throughout which the coefficients α_{ij} and β_i are continuous. We have :

$$\begin{aligned} \sum_{j=1}^{j=n} \alpha_{ij} y_j + \beta_i &= \beta_i + \alpha_{i1} \sum_{k=0}^{k=\infty} y_1^{(k)} + \alpha_{i2} \sum_{k=0}^{k=\infty} y_2^{(k)} + \dots + \alpha_{in} \sum_{k=0}^{k=\infty} y_n^{(k)} \\ &= \beta_i + \sum_{j=1}^{j=n} \alpha_{ij} y_j^{(0)} + \sum_{j=1}^{j=n} \alpha_{ij} y_j^{(1)} + \dots \quad (i = 1, 2, \dots, n). \end{aligned}$$

This last series clearly converges uniformly throughout (J) , since the series (13) do so. It is, however, precisely the series which we should obtain by differentiating (13) term by term. Accordingly y_i has a derivative throughout (J) , and this derivative is represented by the series last written. Moreover, the formula just obtained shows that this derivative satisfies (1). Thus our proof is complete.

It remains merely to prove that two solutions (y_1, \dots, y_n) and $(\bar{y}_1, \dots, \bar{y}_n)$ of (1) which satisfy the conditions

$$y_i(c) = \bar{y}_i(c) \quad (i = 1, 2, \dots, n)$$

must be identically equal. For this purpose we notice that if

$$\eta_i = y_i - \bar{y}_i, \quad (i = 1, 2, \dots, n),$$

then $(\eta_1, \eta_2, \dots, \eta_n)$ is a solution of the system of homogeneous linear equations

$$\frac{d\eta_i}{dx} = \sum_{j=1}^{j=n} a_{ij} \eta_j, \quad (i = 1, 2, \dots, n). \quad (14)$$

Our theorem will, therefore, be established if we can prove the following special case of it:

If $(\eta_1, \eta_2, \dots, \eta_n)$ is a solution of (14), and if

$$\eta_i(c) = 0, \quad (i = 1, 2, \dots, n),$$

then

$$\eta_i(x) \equiv 0, \quad (i = 1, 2, \dots, n).$$

If possible, let h be a point in (I) , where all the functions η_i do not vanish. Then we can, by the portion of theorem I already proved, find $n - 1$ solutions of (14),

$$(\eta_1^{(i)}, \eta_2^{(i)}, \dots, \eta_n^{(i)}) \quad (i = 1, 2, \dots, n - 1)$$

such that the determinant

$$D(x) = \begin{vmatrix} \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-1)} \\ \eta_2 & \eta_2^{(1)} & \dots & \eta_2^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-1)} \end{vmatrix}$$

does not vanish when $x = h$.*

* If, for instance $\eta_n(h) \neq 0$, we may determine these solutions by the conditions:

$$\begin{aligned} \eta_1^{(1)}(h) &= \eta_2^{(2)}(h) = \dots = \eta_{n-1}^{(n-1)}(h) = 1, \\ \eta_i^{(j)}(h) &= 0 \text{ when } i \neq j. \end{aligned}$$

Let us now form the derivative of $D(x)$, writing this derivative as the sum of n determinants, each of which is obtained from D by differentiating all the elements of one row. By a simple reduction, we find

$$D'(x) = (\alpha_{11} + \alpha_{22} + \dots + \alpha_{nn}) D(x).$$

We have, therefore,

$$D(x) = ke^{\int_0^x (\alpha_{11} + \dots + \alpha_{nn}) dx},$$

where k is a constant.* Accordingly, $D(c) = k$. But we obviously have $D(c) = 0$. Accordingly, $D(x) \equiv 0$. This, however, contradicts the fact above mentioned that $D(h) \neq 0$. The assumption that there is a point where all the functions η_i do not vanish has thus led us to a contradiction, and thus our theorem is proved.†

The second fundamental theorem which we wish to prove states, roughly speaking, that a small variation in the coefficients of the system (1) and in the initial values c_i produces only a small variation in the solution. The theorem is this:

II. *The positive constants B , C and ε being given, and also a positive function $\phi(x)$ which has at most a finite number of discontinuities in (I) and for which $\int \phi(x) dx$ converges when extended over any portion of (I) , then a positive δ exists such that if*

$$\begin{aligned} (1) \quad \frac{dy_i}{dx} &= \sum_{j=1}^{j=n} \alpha_{ij} y_j + \beta_i, \\ (\bar{1}) \quad \frac{d\bar{y}_i}{dx} &= \sum_{j=1}^{j=n} \bar{\alpha}_{ij} \bar{y}_j + \bar{\beta}_i, \end{aligned} \quad (i = 1, 2, \dots, n)$$

are any two systems of equations whose coefficients satisfy condition (A) and also (except possibly at a finite number of points) the inequalities:

$$\begin{aligned} |\bar{\alpha}_{ij} - \alpha_{ij}| &\leq \delta, & |\alpha_{ij}| &\leq \phi(x), & |\bar{\alpha}_{ij}| &\leq \phi(x), \\ |\bar{\beta}_i - \beta_i| &\leq \delta, & \left| \int \beta_i dx \right| &\leq B, & \left| \int \bar{\beta}_i dx \right| &\leq B, \end{aligned} \quad (i = 1, 2, \dots, n)$$

* Cf., for example, Sauvage, "Théorie générale des systèmes d'équations différentielles linéaires et homogènes" (1895), p. 13. This thesis was reprinted from the Annales de la Faculté des Sciences de Toulouse, vols. 8 and 9.

† The proof here given is an immediate generalization of the proof of a similar theorem given by Sturm (Liouville's Journal, vol. I (1836), p. 109), for a single linear differential equation of the second order.

(where the last two inequalities are supposed to hold no matter what two points of (I) are taken as the limits of integration), and if c is any point of (I) and (y_1, \dots, y_n) and $(\bar{y}_1, \dots, \bar{y}_n)$ are solutions of (1) and $(\bar{1})$ respectively, for which

$$|\bar{y}_i(c) - y_i(c)| \leq \delta, \quad |y_i(c)| \leq C, \quad |\bar{y}_i(c)| \leq C, \quad (i = 1, 2, \dots, n),$$

then

$$|\bar{y}_i(x) - y_i(x)| < \varepsilon, \quad \left(\begin{matrix} i = 1, 2, \dots, n \\ a \leq x \leq b \end{matrix} \right).$$

To prove this theorem, we express the solutions y_i and \bar{y}_i by the method of successive approximations, as explained above, in the form :

$$\begin{aligned} (3) \quad y_i &= y_i^{(0)} + y_i^{(1)} + y_i^{(2)} + \dots, \\ (\bar{3}) \quad \bar{y}_i &= \bar{y}_i^{(0)} + \bar{y}_i^{(1)} + \bar{y}_i^{(2)} + \dots, \end{aligned} \quad (i = 1, 2, \dots, n).$$

Writing $B + C = M$, we see that every term of (3) and also of ($\bar{3}$) is in absolute value less than or at most equal to the corresponding term of (12).

Let us choose a positive integer l , so that the remainder of (12), after the l^{th} term, is less than $\varepsilon/3$. Then we have :

$$|\bar{y}_i - y_i| < \sum_{k=0}^{l-1} |\bar{y}_i^{(k)} - y_i^{(k)}| + \frac{2\varepsilon}{3}, \quad (i = 1, 2, \dots, n).$$

Our theorem will therefore be proved if we can show that δ (which is as yet wholly unrestricted) can be so chosen that

$$|\bar{y}_i^{(k)} - y_i^{(k)}| < \frac{\varepsilon}{3l}, \quad \left(\begin{matrix} i = 1, 2, \dots, n \\ k = 0, 1, \dots, l-1 \end{matrix} \right).$$

Let us assume that, a positive η being given, a positive δ exists such that

$$|\bar{y}_i^{(k)} - y_i^{(k)}| < \eta, \quad \left(\begin{matrix} i = 1, 2, \dots, n \\ k = 0, 1, \dots, m < l-1 \end{matrix} \right). \quad (15)$$

Then, since we know that such a δ exists, when $k=0$,* we shall have established our theorem by mathematical induction if we can show that by still further decreasing δ (if necessary) (15) can be made to hold when $k=m+1$. For this purpose, let us take δ so that

$$\delta < \frac{\eta \cdot m!}{2nM(nN)^m(b-a)}$$

* We have then merely to take $\delta < \eta/2$, and $\delta < \eta/2(b-a)$.

and then let us still further decrease δ (if necessary), so that :

$$|\bar{y}^{(m)} - y_i^{(m)}| < \frac{\eta}{2nN}, \quad (i = 1, 2, \dots, n).$$

Now we have :

$$\begin{aligned} \bar{y}_i^{(m+1)} - y_i^{(m+1)} &= \int_c^x \sum_{j=1}^{j=n} (\bar{\alpha}_{ij} \bar{y}_j^{(m)} - \alpha_{ij} y_j^{(m)}) dx \\ &= \int_c^x \sum_{j=1}^{j=n} [\bar{\alpha}_{ij} (\bar{y}_j^{(m)} - y_j^{(m)}) + y_j^{(m)} (\bar{\alpha}_{ij} - \alpha_{ij})] dx, \end{aligned} \quad (i = 1, 2, \dots, n).$$

Accordingly,

$$|\bar{y}_i^{(m+1)} - y_i^{(m+1)}| \leq \left| \int_c^x \left(n\phi(x) \cdot \frac{\eta}{2nN} + nM \frac{(nN)^m}{m!} \cdot \frac{\eta \cdot n!}{2nM (nN)^m (b-a)} \right) dx \right| \leq \eta, \quad (i = 1, 2, \dots, n).$$

Thus our theorem is proved.

As an immediate application of the theorem just proved, we mention the following theorem :

If, corresponding to the values $k = 1, 2, 3, \dots$, we have an infinite number of systems of equations

$$\frac{dy_i^{(k)}}{dx} = \sum_{j=1}^{j=n} \alpha_{ij}^{(k)} y_j^{(k)} + \beta_i^{(k)}, \quad (i, j = 1, 2, \dots, n), \quad (16)$$

if the coefficients of each of these systems of equations satisfy conditions (A), and if, in a region consisting of all except a finite number of the points of (I), $\alpha_{ij}^{(k)}(x)$ and $\beta_i^{(k)}(x)$ approach uniformly the functions α_{ij} and β_i as k becomes infinite, and if the constants $c_i^{(k)}$ are such that*

$$\lim_{k=\infty} c_i^{(k)} = c_i, \quad (i = 1, 2, \dots, n),$$

and if (y_1, \dots, y_n) and $(y_1^{(k)}, \dots, y_n^{(k)})$ are solutions of (1) and (16) respectively which satisfy the initial conditions :

$$y_i(c) = c_i, \quad y_i^{(k)}(c) = c_i^{(k)}, \quad (i = 1, 2, \dots, n),$$

then the functions $y_i^{(k)}(x)$ approach uniformly the functions $y_i(x)$ as k becomes infinite.

* It should be noticed that we need not here exclude the possibility that the number of discontinuities of some or all of the functions $\alpha_{ij}^{(k)}$, $\beta_i^{(k)}$ become infinite with k . Cf. the special case mentioned below.

If, in particular, the coefficients of (1) are continuous throughout (I), we may form the equations (16) by dividing the interval (I) into m_k portions, the integer m_k being supposed to increase without limit as k becomes infinite, and all portions into which (I) is divided being supposed to approach zero as their limits. Let us further denote by $x_1^{(k)}, x_2^{(k)}, \dots, x_{m_k}^{(k)}$ points lying one in each of these sub-intervals; and let us define $\alpha_{ij}^{(k)}$ and $\beta_i^{(k)}$ throughout the l^{th} of these intervals by the formulæ:

$$\alpha_{ij}^{(k)}(x) = \alpha_{ij}(x_i^{(k)}), \quad \beta_i^{(k)}(x) = \beta_i(x_i^{(k)}).$$

The coefficients of (16) thus consist, for each value of k , of a succession of constant values, so that the functions $y_i^{(k)}$ can be expressed in terms of elementary functions. The method here sketched for approximating to the solution of (1) was developed for a special case at considerable length by Radaković.*

The results of the present paper can be applied without difficulty to the case of a single linear differential equation of the n^{th} order, since such an equation is equivalent, as is well known, to a system of the form (1).†

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

* Monatshefte für Math. u. Physik, vol. 5 (1894), p. 193.

† The results thus obtained will include, among others, the theorems established by the writer in the Bulletin of the American Mathematical Society for March, 1899, on p. 276.